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Quadratic Function with Fuzziness of a Fixed Point

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Abstract

The fuzzy logic fixed points assist in the examination of system stability in the control systems and clustering domain. They are used in fuzzy equations solving and optimization of fuzzy algorithms. The fixed point theory used in fuzzy systems guarantees convergence and stability as it provides an intermediate between uncertainty (fuzziness) and certain mathematical laws. Combining fuzzy logic with fixed-point theory is a strong mathematical basis of studying systems with uncertainty. Fuzzy fixed points are also useful in analyzing stability in control systems and clustering algorithms, solving fuzzy equations, and optimization of fuzzy algorithms. Fixed-point theory in fuzzy systems, by censoring the distance between uncertainty (fuzziness) and fixed mathematical laws, provides convergence and stability, and provides a more realistic model of the imperfect real world.

The study uses the analytical and numerical methodology in order to study the impact of fuzziness on fixed points in a systematic way. With the introduction of a fuzziness parameter (μ), and operated around a symmetric interval, a classical fixed-point will become a fuzzy mapping. Further breakdown into the functions of linear, quadratic, trigonometric and radical functions will show how fuzziness transforms a deterministic fixed point to a solution interval. The findings indicate that although linear systems have steady and equal interval shifts, nonlinear functions had more complicated and delicate behaviors, such as interval expansion and possible bifurcations. This interval representation of fixed points provides a more useful and more precise representation to use in applications in both control systems and economic modeling and dynamical systems in which parameters are inherently imprecise. The paper concludes by finding classical numerical methods to be still useful, but they need to be reinterpreted where convergence is to a region of solution instead of a point.

Keywords: Fuzzy Fixed Point; Fuzzy Set Theory; Fuzzy Mapping; Fuzzy Analysis

1. Introduction

Fixed-point theory is a branch of mathematical analysis that forms a fundamental basis in the analysis of dynamic systems and the study of nonlinear equations as well as the existence and uniqueness of solutions in different functional spaces [1]. It has extensive applications in many areas such as differential equations, optimization, economics as well as computer science. Simultaneously, fuzzy set theory, which was proposed by Lotfi Zadeh in 1965, transformed the representation of uncertainty and imprecision of the real-world data and processes [2]. It also is more flexible and realistic than classical binary logic as it is based on partial membership with degrees between 0 to 1.

The intersection of these two influential theories gives a fruitful field of research. Deterministic systems have a definite value at a fixed point but when systems are not deterministic then the fixed point is a definite value under the condition of vagueness, noise, or incomplete information--under the concept of fuzziness the idea of fixed point is a natural development. This gives rise to the concept of a fuzzy fixed point which can take the form of an interval, a set of possible values or a membership function that occupies how much a point is fixed under a fuzzy mapping [3]. The dynamics of

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fuzziness in the existence, stability, and properties of fixed points are of vital importance to the development of fuzzy numerical algorithms, robust fuzzy controllers, and to the modeling of complex systems in engineering and the applied sciences [4]. The objective of this paper is to identify this relationship systematically. We shall show how by adding a fuzziness parameter (μ), classical fixed points, which place a single numerical solution, are turned into solution ranges, and thus are more representative of systems running under uncertainty. The analysis will involve linear, quadratic, trigonometric and radical functions to establish the different sensitivity of the various forms of functions to fuzzy perturbations.

2. Methodology

This paper is based on an analytical and numerical methodology of examining the effect of fuzziness on fixed points. The methodology is designed in the following steps:

2.1. Theoretical Foundation

The basic theoretical background is a composite of the classical fixed-point theory of the principles of fuzzy set theory.

Classical Fixed-Point Iteration: The major numerical method is the fixed- point iteration, in which a function g(x) is represented by the recursion equation x_{n+1}=g (x_n). The Banach Fixed-Point Theorem ensures convergence when g is a contraction; that is, when |g 0 (x)| < 1 in some neighborhood of the fixed point [1, 5].

Fuzzy Set Theory: Uncertainty is added through an element of fuzziness, μ, which lies in a given symmetric interval $[-\alpha, \alpha]$. This aspect is introduced into the classical expression so as to form a fuzzy mapping, $f_\mu(x) = f(x) + \mu$, which is imprecision within a limit of the model of the system [2, 6].

3. Modelling Fuzzy Fixed Points

Given a classical function, f (x) we define the fuzzy fixed-point equation as:

$$x = f(x) + \mu, \qquad \mu \in [-\alpha, \alpha].$$

The problem will be to find x as a function of μ. The set of solutions of all the μ in its.

interval gives a definition of the fuzzy fixed point that is usually an interval [x_{min}, x_{max}].

4. Analytical and Numerical Process.

The steps to be followed when the type of the function (linear, quadratic, etc.) are to be used are the following:

- Formulation: Stated classical fixed-point equation x = f (x) and find its solution.
- Fuzzification: Add the parameter μ to get x = f (x) + μ.
- Solution obtaining an analytic solution: Obtaining an analytic solution obtaining an analytic solution is possible, when the equation to be solved is a linear function. In nonlinear cases (e.g. x = cos (x) + μ), the classical fixed point (μ =0) is first approximated numerically by iterative methods [5].
- Interval Determination: Find the limits of the solution x by computing the expression of the two extreme cases $\mu = -\alpha$ and $\mu = +\alpha$.
- Analysis: Compare the classical fixed point and the fuzzy fixed-point interval commenting on the shift, fixed point stability and sensitivity.

This methodology is used continuously in four different types of functions to allow them to make comparative analysis of the way fuzziness interacts with various mathematical structures.

5. Fixed Point: Concept and Theoretical Basis.

5.1. What is a Fixed Point?

The fixed point of any function f is a value of the domain of f at which f (x) = x. That is, when the point is applied, the point is not changed.

Example: Consider f (x) = x^2, so that the fixed points of this problem are x = 0 and x = 1.

Example: In the case of f (x) = (x + 2)/3, the fixed point is found by solving x = (x + 2)/3, x = 1.

5.2. Fixed-Point Iteration

It is a numerical technique of solving equations of the type x = g(x).

Steps:

- Reform the original equation f (x) = 0 in the form of x = g(x).
- Choose an initial guess x_0.
- Iterate $g(x_n) = x_{n+1}$ until convergence.

5.3. Convergence Criterion (Banach Theorem):

In case the function g is a contraction, that is, |g 1(x) | if $|g'(x)| < 1$ in the intervals between which the fixed point lies, i.e., in [1, 5], the iteration will converge to a single fixed point.

Examples of Choosing g(x):

For $x^2 - x - 1 = 0$: g(x)= (√x+1)

(Convergent, as $|g'(1.618)| \approx 0.309 < 1$).

g(x) = x2 − 1 (Divergent, as $|g'(1.618)| \approx 3.236 > 1$).

For *cos(x) = x:*

$g(x) = \cos(x)$ (Converges to $x \approx 0.739085$, with $|-\sin(0.739085)| \approx 0.673 < 1$).

5.4. Practical Considerations:

- The convergence rate is linear and it will be determined by $|g'(x)|$ screams.
- Various rearrangements of f (x) = 0 result in various iteration functions having different convergence properties.
- There can be a number of fixed points and the starting guess will decide which point to approach. Fixed-point iteration is more time-consuming than Newton-Raphson (quadratic convergence), yet it does not need the evaluation of derivatives [5].

6. Fuzziness and Fuzzy Logic

6.1. The Concept of Fuzziness

Fuzziness concerns itself with uncertainty or vagueness in which membership of a set is not a decision between 0 and 1 but is a degree of choice, and can take any value between 0 and 1 [2].

6.2. Fuzzy Logic

Fuzzy logic is a many-valued logic that manages approximate logical thinking.

- Fuzzy Sets: Permit the gradual membership non-membership.
- Membership Functions: Measures the level of the membership of an element in a fuzzy set.
- Linguistic Variables: Words (e.g. hot, cold) are values of the variable.
- Fuzzy Rules: Rules that are based on inference are referred to as IF-THEN rules.
- Defuzzification: transforms the fuzzy output into a hard and actuarial value.

Use: Control systems (washing machines, air conditioners), decision- making in uncertain scenarios, and artificial intelligence [6, 7].

7. Relating Fuzziness and Fixed-Point Theory.

The concepts of fixed-point are applied to fuzzy logic in various manners [3, 4, 8]:

1.Fixed Points in Fuzzy Sets: In using fuzzy operations Fuzzy operations (union, intersection, complement) have fixed points in which membership degree is not changed under the operation under the operation. As an example, in case of fuzzy negation:

$\mu A(x) = 1\text{-}\mu A(x)$ implies.

$\mu A(x) = 0.5$.

2.Fixed points in Fuzzy inference systems: In Mamdani or Sugeno systems, fixed- point analysis is used to find equilibrium states of repeated application of fuzzy rules, which is important in stable fuzzy control systems.

3. The Banach Fixed-Point Theorem in Fuzzy Metric Spaces: Distance measured by using fuzzy sets, or other similar spaces, has generalized the classical version to such spaces membership functions, which are made to cover fuzzy mappings [8, 9].

7.1. Computational Exempla Classical vs. Fuzzy Fixed points.

7.1.1. Example 1: Linear Function

Classical Equation:

$f(x) = 0.6x + 2$

Fixed point: $x = 0.6x + 2 \rightarrow 0.4x = 2 \rightarrow x = 5$.

With Fuzziness ($\mu \in [-0.2, 0.2]$):

$f\mu(x) = 0.6x + 2 + \mu$

Fuzzy fixed point: $x = 0.6x + 2 + \mu \rightarrow 0.4x = 2 + \mu \rightarrow x = (2 + \mu)/0.4$

For $\mu \in [-0.2, 0.2]$: $x \in [(1.8)/0.4, (2.2)/0.4] \rightarrow x \in [4.5, 5.5]$.

7.1.2. Example 2: Quadratic Function

Classical Equation: $f(x) = x^2 - 2x + 2$

With Fuzziness ($\mu \in [-0.2, 0.2]$): $f\mu(x) = x^2 - 2x + 2 + \mu$

Fixed point: $x = x^2 - 2x + 2 + \mu$

Rearranging: $x^2 - 3x + (2 + \mu) = 0$

Applying the quadratic formula: $x = (\sqrt{1 - 4\mu})/2$

Calculating the range:

For $\mu = -0.2$: $\approx \sqrt{1.34} \rightarrow x \approx [0.83, 2.17]$

For $\mu=0.2$: $\approx \sqrt{0.2} \rightarrow x \approx [1.275, 1.725]$

The rough approximate overall fuzzy interval: x [0.83, 2.17].

7.1.3. Example 3: Trigonometric (Cosine) Function

Can be used to represent the concept of regression as demonstrated in the following example.

Classical Equation: $f(x) = \cos(x)$

Fixed point (Dottie number): $x \approx 0.739085$.

With Fuzziness ($\mu \in [-0.05, 0.05]$): $f\mu(x) = \cos(x) + \mu$

Fuzzy fixed point: $x = \cos(x) + \mu$

By taking the classical solution as reference:

$x \in [\cos(0.739085) - 0.05, \cos(0.739085) + 0.05] = [0.739085 - 0.05, 0.739085 + 0.05]$

$x \in [0.689085, 0.789085]$.

7.1.4. Example 4: Square Root Function

Classical Equation: $f(x) = (\sqrt{X}+3)$

Fixed point: $x=(\sqrt{X}+3)$ $x = 2.3028$.

With Fuzziness ($\mu \in [-0.1, 0.1]$): $f\mu(x) = (\sqrt{X}+3)+\mu$

Fuzzy fixed point: $x=(\sqrt{X}+3)+\mu$

The square of this equation gives a quadratic, which on solution leads to an approximation interval:

$x \in [2.15, 2.43]$.

8. Results and Discussion

- Fixed-Point Nature Transformation: The idea of fuzziness was always used to convert a classical fixed point of one numerical value into a range or interval of potential values [3, 10]. This is in line with the common sense anticipation that the solution set should reflect system uncertainty, by giving a more realistic model of imperfect real-world systems.
- Effects of Fuzziness on Stability: Stability and even existence of a fixed point will be conditional on the size and form of the fuzziness parameter μ [4, 9]. Although small perturbations ($|\mu|$ small) only tend to create a bounded change in the fixed point location, greater fuzziness can much more drastically change the mathematical landscape. With nonlinear functions (Examples 2 and 4), the discriminant may be influenced by fuzziness, which may lead to fixed points or fixed points being removed or even invalid (e.g. extraneous solutions due to squaring).
- Type Differentiated Sensitivity of Function types: As shown in the results, the sensitivity types are clearly varied:
    - Linear Functions (Example 1): The fixed point is assured to remain and just changes linearly with μ and lead to a symmetric interval. It means that linear systems are intrinsically fuzzy stable.
    - Nonlinear Functions (Examples 2-4): These are more sensitive and found to have more complicated behavior.

A broadening of the solution interval and possible bifurcation-like behavior on the basis of the discriminant was observed with quadratic functions.

In the square-root, the absence of extraneous solutions necessitated such care in manipulating algebraic expressions that this becomes a procedure of some sensitiveness in computing fuzzy radicals.

The trigonometric function retained a convergent steady convergent behavior yet the exact equilibrium value was fuzzy.

- Better Real World Embodiment: Fuzzy fixed points, in the form of intervals, provide a more useful and informative representation of applications like control.

processes, economic modelling, and any dynamical system whose parameters cannot be known at infinite precision [6, 7].

5.Iterability of the Classical methods of fixed-point iteration Techniques of classical fixed-point iteration can also be used to approximate fuzzy fixed points. However, the interpretation changes: the iteration sequence will converge towards a region between the fuzzy interval [5, 10] rather than to a number. Stopping criteria and error analysis have to be modified to take into consideration this inherent solution width.

9. Conclusions

- The nature of fixed points is fundamentally changed under fuzziness and they become fixed points, makes deterministic values sets or intervals, which is more precise in the case of systems performing in the uncertain environment.
- The fuzziness parameter amplitude and shape of the structure of the governing function is vital in determining stability and properties of a fuzzy fixed point.
- It is also seen that nonlinear functions exhibit much more sensitive behavior and behavioral changes in fixed points at levels of fuzziness being much more complex in the nonlinear case than they are in the case with linear functions.
- Conventional numerical techniques still are tools that are still useful, but must be modified interpretation in the case of fuzzy equations, convergence being taken to mean approach to a solution region.
- Their combination with fixed-point theory produces a strong mathematical framework of study of the stability and equilibrium of complex systems with uncertainty conditions with a broad potential in application to engineering, economics, and computational sciences.

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