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(Research Article)

On Jordan (ϭ, τ)-k-Homomorphism of Γ-rings

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Abstract

Suppose L a Γ-ring and Ḿ be a Γ́ -ring . I n the presented paper introduced the concept of (ϭ,τ)-k-homomorphism , Jordan (ϭ,τ)-k-homomorphism, Jordan triple (ϭ,τ)- k homomorphism (ϭ,τ)-k - anti homomorphism and state under certain settings of L that each Jordan (ϭ,τ)-k-homo-morphism θ from a Γ-ring L to a 2-torsion free Γ́ -ring Ḿ , for example ($x\alpha y\beta y$) = ($x\beta y\alpha x$), where ($x, y \in$ L) and ($\alpha, \beta \in \Gamma$), {x'y'β'x'} = {x'β'b'x'x'}, where (x', y' ∈ Ḿ) and (α', β'∈ Γ́) , ($ϭ^2$ = ϭ, τ^2 = τ and ϭ τ = τ ϭ), then θ is Jordan triple (ϭ ,τ)-k-homo-morphism.

Keywords: Prime Γ– ring; Homomorphism; Jordan homomorphism; Endomorphisms; Mapping; Triple

1. Introduction

Assume L and Γ are two additive sets, with mapping between L and Γ. L×Γ×L⟶ L copy of (x,α,y) existence symbolized by ($x\alpha y$, $x, y \in$ L and $\alpha \in \Gamma$) , satisfying the these conditions:

If ($x, y, z \in$ L) and ($\alpha, \beta \in \Gamma$):

(i) $(x + y)\, \alpha z = x\alpha z + y\alpha z$

$x(\alpha + \beta)\, z = x\alpha z + x\beta z$

$x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha y\beta z$

In this case, L is a Γ-ring. This definition is attributed to Barnes [1].

A Γ-ring L is named a prime if (xΓLΓy) = (0) requiring x = 0 or y = 0, where $x, y \in$ L)[2].

A Γ-ring L is named semi prime if xΓLΓy = (0) denotes x = 0, such that $x\in$ L [2] .

Suppose that L is a semi prime 2-torsion free. Γ-rig. Assume that $x, y \in$ L, when xΓmΓy + yΓmΓx = 0 , m ∈ L. Then xΓmΓy = yΓmΓx = 0 [3] .

Let L be Γ-ring, it said to be 2- torsion free when 2a = 0 then denotes a = 0, where $a \in$ L [4,5,6].

The addition mapping θ of a Γ-ring L to a Γ-ring Ḿ named homomorphism when θ($x\alpha y$) = θ(x)αθ(y), (x, y) ∈ L and (α) ∈ Γ [1].

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The addition mapping θ of Γ-ring L to a Γ-ring Ḿ, named Jordan homomorphism, when $\theta(x\alpha y + y\alpha x) = \theta(x)\ \alpha\theta(y) + \theta(y)\ \alpha\theta(x)$ $(x, y) \in$ L and $(\alpha \in \Gamma)$ [5,6].

Also θ of a Γ-ring L to a Γ-ring Ḿ and σ, τ be two endomorphisms of L, θ named (σ,τ)-homomorphism ,when $\{\theta(x\alpha y) = \theta(\sigma(x))\alpha\theta(\tau(y))\}$, $x, y \in$ L and $\alpha \in \Gamma$ [2,3,4].

In addition, θ of a Γ-ring L to a Γ-ring Ḿ and σ, τ be two endo-morphisms of L, θ called Jordan (σ ,τ)-homomorphism ,when $\theta(x\alpha y + y\alpha x) = \theta(\sigma(x))\ \alpha\theta(\tau(y)) + \theta(\sigma(y))\ \alpha\theta(\tau(x))$, (x, y) ∈ L and α ∈ Γ [2,3,4].

The essential aim for this work, is to explain a Jordan (σ ,τ)- k - homomorphism of a Γ-ring L to prime Γ́-ring Ḿ is either (σ ,τ)-k-homomorphism or (σ ,τ)-k -antihomomorphism.

2. (σ,τ)-k-Homomorphism of Γ-rings

2.1. Definition

Assume θ addition map of a Γ-ring L to a Γ́-ring M' and K be an addition map of Γ into Γ́, σ,τ be two endomorphisms of L, θ is called (σ,τ)- k-homomorphism when $\theta(x\alpha y) = \theta(\sigma(x))k(\alpha)\theta(\tau(y))$, $x, y \in$ L and $\alpha \in \Gamma$.

2.2. Example

Consider R, as ring . Take L = $L_{1\times 2}(R)$, Ḿ = $L_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in Z \right\}$. This means both L and Ḿ are two Γ-rings.

Let $k=(k_i)_{i\in N}$ be a group of maps, collectively from and to itself .
Assume θ addition mapping of a Γ-ring L to a Γ-ring Ḿ, for example: $\theta(x\ y) = (-x\ 0)$, $(x\ y) \in$ L. Assume σ, τ be two endomorphisms of L , that satisfies:

$\sigma(x\ y)=(x\ 0)$, $\tau(x\ y)=(-x\ y)$. Then θ is (σ ,τ)-k-homo-morphism [5,6,7].

2.3. Definition

Suppose θ as addition map of a Γ-ring L to a Γ́ -ring Ḿ and K be additive mapping of Γ into Γ́ , σ,τ be two endomorphisms of L, θ called Jordan (σ,τ)- k –homo-morphism, provided that

$\theta(x\alpha y + y\alpha x) = \theta(\sigma(x))k(\alpha)\theta(\tau(y)) + \theta(\sigma(y))k(\alpha)\theta(\tau(x))$ (x, y) ∈ L and α ∈ Γ.

2.4. Definition

When θ is additive mapping of a Γ-ring L to a Γ́ -ring Ḿ and K be an addition map of Γ into Γ́, σ,τ be two endomorphisms of L, θ icalled Jordan triple (σ,τ)- k–homo-morphism when

$\theta(x\alpha y\beta x)=\theta(\sigma(x))\ k(\alpha)\ \theta(\sigma\ \tau(y))\ k(\beta)\ \theta(\tau(x))$, (x, y) ∈ L and α, β ∈ Γ.

2.5. Definition

Consider θ addition map of a Γ-ring L into a Γ́-ring Ḿ and K be additive mapping of Γ into Γ́ , σ , τ be two endomorphisms of L, θ called (σ,τ)-k - anti homomorphism, when $\theta(x\alpha y) = \theta(\sigma(y))\ k(\alpha)\ \theta(\tau(x))$, $x, y \in$ L and $\alpha \in \Gamma$.

2.6. Lemma

Consider θ as Jordan triple (σ,τ)-k-homomorphism from Γ-ring L to a Γ́ -ring Ḿ , if x, y, z ∈ L and α, β ∈ Γ

(i) Let $\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma\tau = \tau\sigma$,
$(x\beta y\alpha x) + \theta(x\alpha y\beta x) = \theta\sigma(x)\ k(\beta)\ \theta\sigma\tau(y)\ k(\alpha)\ \theta\tau(y) + \theta\sigma(x)\ k(\alpha)\ \theta\sigma\ \tau(y)\ k(\beta)\ \theta(\tau(x))$

(ii) $(z\alpha y\beta x) + \theta(x\alpha y\beta z) = \theta\sigma\ (z)\ k(\alpha)\ \theta\sigma\tau(y)\ k(\beta)\ \theta\tau(x) + \theta\sigma(x)\ k(\alpha)\ \theta\sigma\ \tau(y)\ k(\beta)\ \theta\tau(z)$
(iii) Take L, Ḿ be two alternate and Ḿ is a 2-torsion free , then : $\theta(x\alpha y\beta z) = \theta\sigma\ (x)\ k(\alpha)\ \theta\sigma\ \tau(y)\ k(\beta)\ \theta(\tau(z))$
(iv) $\theta(x\alpha y\alpha z) + (z\alpha y\alpha x)=\theta(\sigma\ (x))\ k(\alpha)\ \theta(\sigma\tau(y))\ k(\alpha)\ \theta(\tau(z)) + \theta(\sigma\ (z))\ k(\alpha)\ \theta(\sigma\ \tau(y))\ k(\alpha)\ \theta(\tau(x))$

Proof

(i) Instead of $x\beta y + y\beta x$ for y in Definition (2.3), this result will be,
$\theta(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = \theta(\sigma(x))k(\alpha)\theta(\tau(x\beta y + y\beta x)) + \theta(\sigma(x\beta y + y\beta x))k(\alpha)\theta(\tau(x))$

$= \theta(\sigma(x))k(\alpha)\theta(\tau(x)\beta\tau(y)+ \tau(y)\beta\tau(x)) + \theta(\sigma(x)\beta\sigma(y) + \sigma(y)\beta\sigma(x))k(\alpha)\theta(\tau(x))$

$= \theta(\sigma(x))k(\alpha)\theta(\sigma\tau(x))k(\beta)\theta(\tau^2(y)) + \theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau^2(x)) + \theta(\sigma^2(x))k(\beta)\theta(\tau\sigma(y))k(\alpha)\theta(\tau(x)) + \theta(\sigma^2(y))k(\beta)\theta(\tau\sigma(x))k(\alpha)\theta(\tau(x))$

Since $\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma\tau = \tau\sigma$

$= \theta(\sigma(a))k(\alpha)\theta(\sigma\tau(a))k(\beta)\theta(\tau(b)) + \theta(\sigma(a))k(\alpha)\theta(\sigma\tau(b))k(\beta)\theta(\tau(a)) + \theta(\sigma(a))k(\beta)\theta(\sigma\tau(b))k(\alpha)\theta(\tau(a)) + \theta(\sigma(b))k(\beta)\theta(\sigma\tau(a))k(\alpha)\theta(\tau(a))$

……………..….(1)

On the other side

$(x\beta y + y\beta x)\alpha x) + \theta(x\alpha(x\beta y + y\beta x) = \theta(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)$
$=\theta\sigma(x)k(\alpha)\theta\sigma\tau(x)k(\beta)\theta\tau(y)+\theta\sigma(y)k(\beta)\theta\sigma\tau(x)k(\alpha)\theta\tau(x)+\theta(x\alpha y\beta x+ x\beta y\alpha x)$ ……………….…..(2)

The comparison between (1) and (2), will lead to,

$\theta(x\alpha y\beta x+x\beta y\alpha x)=\theta\sigma(x)k(\alpha)\theta\sigma\tau(y)k(\beta)\theta\tau(y)+\theta\sigma(x)k(\beta)\theta\sigma\tau(y)k(\alpha)\theta\tau(x)$

(ii) Let us put $x + z$ for x in Definition (2.4), then this will lead to :
$\theta((x + z)\alpha y\beta(x + z)) = \theta(\sigma(x + z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x + z))$

$= \theta(\sigma(x) + \sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x) + \tau(z))$

$= \theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x))+\theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(z)) +$

$\theta(\sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x))+\theta(\sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(z))$

…………….…(3)

On the other hand

$\theta((x + z)\alpha b\beta(x + z)) = \theta(x\alpha b\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z)$

$=\theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x))+\theta(\sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(z))+\theta(x\alpha y\beta z+z\alpha y\beta x)$ ………………..…(4)

By comparing (3) with (4), the following result will be:

$\theta(x\alpha y\beta z+ z\alpha y\beta x) = \theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(z)) + \theta(\sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(x))$

(iii) Based on part (ii) and both L, Ḿ two alternate and Ḿ is 2-torsion free .The result is
$\theta(x\alpha y\beta z + x\alpha y\beta z) = 2\theta(x\alpha y\beta z) = \theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\beta)\theta(\tau(z))$

(iv) Let β instead of α in branch (ii),$\theta(x\alpha y\alpha z+z\alpha y\alpha x)=\theta(\sigma(x))k(\alpha)\theta(\sigma\tau(y))k(\alpha)\theta(\tau(z))+\theta(\sigma(z))k(\alpha)\theta(\sigma\tau(y))k(\alpha)\theta(\tau(x))$

2.7 Definition

Suppose θ Jordan (σ,τ)-k-homo-morphism of a Γ-ring L to a Γ́-ring Ḿ,

$x,y\in$L and $\alpha\in\Gamma$. Let us define G: $L\times\Gamma\times L \longrightarrow$ Ḿ by $G(x,y)_\alpha = \theta(x\alpha y) - \theta(\sigma(x))k(\alpha)\theta(\tau(y))$

2.7. Lemma

Suppose θ Jordan (ϭ,τ)-k-homo-morphism of a Γ-ring L to a Γ́-ring Ḿ. Now if (x, y, z) ∈ L and α, β ∈ Γ.

(i) $G(x, y)_\alpha = -G(y, x)_\alpha$

(ii) $G(x+y, z)_\alpha = G(x,z)_\alpha + G(y,z)_\alpha$

(iii) $G(x, y+z)_\alpha = G(x,y)_\alpha + G(x,z)_\alpha$

(iv) $G(x,y)_{\alpha+\beta} = G(x,y)_\alpha + G(x,y)_\beta$

Proof

(i) Let us depend on Definition 2.3.

$\theta(x\alpha y + y\alpha x) = \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y)) + \theta(\sigma(y))\, k(\alpha)\, \theta(\tau(x))$

$\theta(x\alpha y) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y) = -(\theta(y\alpha x) - \theta(\sigma(y))k(\alpha)\theta(\tau(x))$

$G(x,y)_\alpha = -G(y,x)_\alpha$

(ii) $G(x+y,z)_\alpha = \theta((x+y)\alpha z) - \theta(\sigma(x+y))\, k(\alpha)\theta(\tau(z))$

$= \theta(x\alpha z + y\alpha z) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(z)) - \theta(\sigma(y))\, k(\alpha)\, \theta(\tau(z))$

$= \theta(x\alpha z) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(z)) + \theta(y\alpha z) - \theta(\sigma(y))\, k(\alpha)\theta(\tau(z))$

$= G(x,z)_\alpha + G(y,z)_\alpha$

(iii) $G(x,y+z)_\alpha = \theta(x\alpha(y+z)) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y+z))$

$= \theta(x\alpha y + x\alpha z) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y)) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(z))$

$= \theta(x\alpha y) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y)) + \theta(x\alpha z) -$

$\theta(\sigma(x))\, k(\alpha)\, \theta(\tau(z)) = G(x,y)_\alpha + G(x,z)_\alpha$

(iv) $G(x,y)_{\alpha+\beta} = \theta(x(\alpha+\beta)y) - \theta(\sigma(x))\, k(\alpha+\beta)\, \theta(\tau(y))$

$= \theta(x\alpha y) - \theta(\sigma(x))\, k(\alpha)\, \theta(\tau(y)) + \theta(x\beta y) - \theta(\sigma(x))\, k(\beta)\, \theta(\tau(y))$

$= G(x,y)_\alpha + G(x,y)_\beta$

2.8. Remark

The necessary and sufficient that θ (ϭ,τ)-k-homomorphism of a Γ-ring L into a Γ́-ring Ḿ, is $\{G(x,y)_\alpha = 0$, where $x, y \in L$ and $\alpha \in \Gamma\}$.

2.9. Lemma

Assume θ Jordan (ϭ,τ)-k-homomorphism of a Γ-ring L into a Γ́-ring L,

such that $\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma\tau = \tau\sigma$, where $x, y, n \in L$ and $\alpha, \beta \in \Gamma$

(i) $G(\sigma(x)\,\sigma(y))_\alpha\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, G(\tau(y)\,\tau(x))_\alpha + G(\sigma(y)\,\sigma(x))_\alpha\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, G(\tau(x)\,\tau(y))_\alpha = 0$
(ii) $G(\sigma(x)\,\sigma(y))_\alpha\, k(\alpha)\, \theta(\sigma\tau(n))\, k(\alpha)\, G(\tau(y)\,\tau(x))_\alpha + G(\sigma(y)\,\sigma(x))_\alpha\, k(\alpha)\, \theta(\sigma\tau(n))\, k(\alpha)\, G(\tau(x)\,\tau(y))_\alpha = 0$
(iii) $G(\sigma(x)\,\sigma(y))_\beta\, k(\alpha)\, \theta(\sigma\tau(n))\, k(\alpha)\, G(\tau(y)\,\tau(x))_\beta + G(\sigma(y)\,\sigma(x))_\beta\, k(\alpha)\, \theta(\sigma\tau(n))\, k(\alpha)\, G(\tau(x)\,\tau(y))_\beta = 0$

Proof

(i) Let $w = x\alpha y\beta n\beta y\alpha x + y\alpha x\beta n\beta x\alpha y$, due to θ is a Jordan (σ,τ)-k-homo-morphism

$\theta(w) = y\alpha(x\beta n\beta x)\alpha y) + \theta(x\alpha(y\beta n\beta y)\alpha x$

$= \theta\sigma(x)\, k(\alpha)\, \theta\sigma\tau(y\beta n\beta y)\, k(\alpha)\theta\tau(x)+$

$\theta\sigma(y)\, k(\alpha)\, \theta\sigma\tau(x\beta n\beta x)\, k(\alpha)\theta\tau(y)$

$= \theta\sigma(x)\, k(\alpha)\, \theta\sigma(\sigma\tau(y))\, k(\beta)\, \theta\sigma\tau(\sigma\tau(n))\, k(\beta)\, \theta\tau(\sigma\tau(y))\, k(\alpha)\, \theta\tau(x) +\theta\sigma(y)\, k(\alpha)\, \theta\sigma(\sigma\tau(x))\, k(\beta)\, \theta\sigma\tau(\sigma\tau(n))\, k(\beta)$

$\theta\tau(\sigma\tau(x))\, k(\alpha)\theta\tau(y)$ ……………………(5)

On the other hand

$\theta(w) = \theta((x\alpha y)\, \beta n\beta(y\alpha x) + (y\alpha x)\, \beta n\beta(x\alpha y))$

$= \theta(\sigma(x\alpha y))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, \theta(\tau(y\alpha x))$

$+\theta(\sigma(y\alpha x))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, \theta(\tau(x\alpha y))$

$= \theta\sigma\, (x\alpha y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)(\theta\sigma\tau(x))\, k(\alpha)\, \theta\tau^2(y)$

$+\theta\sigma\tau(y)\, k(\alpha)\, \theta\tau^2(x) - \theta\tau(x\alpha y) + (-\theta(\sigma(x\alpha y))+ \theta(\sigma^2(x))\, k(\alpha)\, \theta(\tau\sigma(y))+$

$\theta(\sigma^2(y))\, k(\alpha)\, \theta(\tau\sigma(x)))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, \theta(\tau(x\alpha y))$

$= -\{\theta(\sigma(x\alpha y))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, (\theta(\tau(x\alpha y)) - \theta(\sigma\tau(x))\, k(\alpha)\, \theta(\tau^2(y))) -$

$\theta(\sigma(x\alpha y))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, (\theta(\tau(x\alpha y)) - \theta(\sigma\tau(y))\, k(\alpha)\, \theta(\tau^2(x))) +$

$\theta(\sigma^2(x))\, k(\alpha)\, \theta(\tau\sigma(y))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, \theta(\tau(x\alpha y))+$

$\theta(\sigma^2(y))\, k(\alpha)\, \theta(\tau\sigma(x))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, \theta(\tau(x\alpha y)$ …………………… (6)

Compare (5), (6) and since $\sigma\tau = \tau\sigma$

$0 = -\theta\sigma(x\alpha y)\, k(\beta)\theta\sigma\tau(n)k(\beta)\, G(\tau(y)\tau(x))_\alpha - \theta\sigma\, (x\alpha y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(y)\tau(x))_\alpha + \theta\sigma^2(x)\, k(\alpha)\theta\tau\sigma(y)\, k(\beta)\theta\sigma\tau(n)$

$\theta\tau(x\alpha y)k(\beta)+ \theta\sigma^2(y)k(\alpha)\theta\tau\sigma(x)\, (\beta)\theta\sigma\tau(n)k(\beta)\theta\tau(x\alpha y)\theta\sigma(x)k(\alpha)\theta\sigma^2\tau(y)k(\beta)\theta\sigma^2\tau^2(n)\, k(\beta)\theta\sigma\tau^2(y)$

$k(\alpha)\theta(\tau(x))\theta(\sigma(y))k(\alpha)\theta(\sigma^2\tau(x))k(\beta)\theta(\sigma^2\tau^2(n))k(\beta)\theta(\sigma\tau^2(x))k(\alpha)\theta(\tau(y)$

While $\sigma^2 = \sigma$, $\tau^2 = \tau$

$0 = -\theta\sigma(x\alpha y)\, k(\beta)\theta\sigma\tau(n)\, k(\beta)\, G(\tau(y)\tau(x))_\alpha - \theta\sigma(x\alpha y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(y)\, \tau(x))_\alpha + \theta\sigma\, (x)\, k(\alpha)\, \theta\tau\sigma\, (y)\, k(\beta)\theta\sigma\, \tau(n)\, k(\beta)\, \theta\tau(x\alpha y) + \theta\sigma\, (y)\, k(\alpha)\, \theta\tau\sigma\, (x)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, \theta\tau\, (x\alpha y) - \theta\sigma\, (x)\, k(\alpha)\, \theta\tau\sigma\, (y)\, k(\beta)\, \theta\sigma(y)\, k(\alpha)\, \theta\tau\sigma(x)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, \theta\sigma\tau\, (x)\, k(\alpha)\, \theta\tau(y)$

$0 = -\theta(\sigma(x\alpha y))\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, G(\tau(x)\, \tau(y))_\alpha - \theta(\sigma(x\alpha y))\, k(\beta)\, \theta(\sigma\, \tau(n))\, k(\beta)\, G(\tau(y)\, \tau(x))_\alpha + \theta(\sigma(x))\, k(\alpha)\, \theta(\tau\sigma(y))$

$k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, (\theta(\tau(x\alpha y)) - \theta\sigma\, \tau(y)\, k(\alpha)\, \theta\tau(x) + \theta\sigma(y)\, k(\alpha)\, \theta\tau\sigma\, (x)\, k(\beta)\, \theta\sigma\, \tau(n)\, k(\beta)\, \theta\tau(x\alpha y) - \theta\sigma\, \tau(x)\, k(\alpha)$

$\theta\tau(y)$

$0 = -\theta\sigma(x\alpha y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} - \theta\sigma(x\alpha y)\, k(\beta)\, \theta\sigma\tau\, (n)\, k(\beta)\, G(\tau(y)\, \tau(x)_{\alpha} +$

$\theta\sigma(x)\, k(\alpha)\, \theta\tau\sigma\, (y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(y)\, \tau(y)_{\alpha} + \theta\sigma(y)\, k(\alpha)\, \theta\tau\sigma\, (x)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(x)\, \tau(y)_{\alpha}$

$0 = -\theta\sigma(x\alpha y) - \theta\sigma(y)\, k(\alpha)\, \theta\tau\sigma\, (x)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} -$

$\theta\sigma(x\alpha y) - \theta\sigma(x)\, k(\alpha)\, \theta\tau\sigma\, (y)\, k(\beta)\, \theta\sigma\tau(n)\, k(\beta)\, G(\tau(y)\, \tau(x)_{\alpha}$

Thus, we have:

$G(\sigma\, (x)\sigma\, (y)_{\alpha}\, k(\beta)\, \theta(\sigma\tau(n))\, k(\beta)\, G(\tau(y)\, \tau(x))_{\alpha} +$

$G(\sigma\, (y)\, \sigma\, (x))_{\alpha}\, k(\beta)\, \theta(\sigma\, \tau(n))\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} = 0$

(ii) Put β instead α in branch (i) and following same steps as in (i), we will obtain the result (ii).
(iii) Now we can put α and β in (i), the result will be (iii).

2.10. Lemma

Consider θ as Jordan (σ,τ)-k-homomorphism of a Γ-ring L to a 2- torsion free prime Γ́-ring Ḿ, if x, y, m ∈ L and α, β ∈ Γ

(i) $G(\sigma\, (x)\, \sigma(y))_{\alpha}\, k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(y)\, \tau(x))_{\alpha} = G(\sigma(y)\, \sigma(x))_{\alpha}\, k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} = 0$
(ii) $G(\sigma(x)\, \sigma(y))_{\alpha}\, k(\alpha)\, \theta(\sigma\tau(m))\, k(\alpha)\, G(\tau(y)\, \tau(x))_{\alpha} = G(\sigma(y)\, \sigma(x))_{\alpha}\, k(\alpha)\, \theta(\sigma\tau(m))\, k(\alpha)\, G(\tau(x)\, \tau(y))_{\alpha} = 0$
(iii) $G(\sigma(x)\, \sigma(y))_{\beta}\, k(\alpha)\, \theta(\sigma\tau(m))\, k(\alpha)\, G(\tau(y)\, \tau(x))_{\beta} = G(\sigma(y)\, \sigma(x))_{\beta}\, k(\alpha)\, \theta(\sigma\tau(m))\, \alpha G(\tau(x)\, \tau(y))_{\beta} = 0$

Proof

(i) According to Lemma (2.10)
$G(\sigma(x)\, \sigma\, (y))_{\alpha}\, k(\beta)\, \theta(\sigma\, \tau(m))\, k(\beta)\, G(\tau(y)\, \tau(x))_{\alpha} + G(\sigma(y),\sigma(x))_{\alpha} k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} = 0$

and on the basis of the lemma (assume M 2-torsion free semi prime Γ-ring and $(x, y)\in$M, if $x\Gamma.M\Gamma y + y\Gamma.M\Gamma x = 0$, then $x\Gamma.M\Gamma y = y\Gamma.M\Gamma x = 0$. The result is,

$G(\sigma(x)\, \sigma(y))_{\alpha}\, k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(y)\, \tau(x))_{\alpha} =$

$G(\sigma(y)\, \sigma(x))_{\alpha}\, k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(x)\, \tau(y))_{\alpha} = 0$

(ii) Instead of α , put β in part (i), to lead to (ii).
(iii) Each of α and β are replaced in (i), aiming to (iii).

2.11. Lemma

Let θ be a Jordan (σ,τ)-k-homo-morphism of a Γ-ring M into a prime Γ́-ring Ḿ, if (x, y, z, w, m) ∈ M and α, β ∈ Γ, then

(i) $G(\sigma\, (x)\, \sigma\, (y))_{\alpha}\, k(\beta)\, \theta(\sigma\tau(m))\, k(\beta)\, G(\tau(w)\, \tau(z))_{\alpha} = 0$
(ii) $G(\sigma\, (x)\, \sigma\, (y))_{\alpha}\, k(\alpha)\, \theta(\sigma\tau(m))\, k(\alpha)\, G(\tau(w)\, \tau(z))_{\alpha} = 0$
(iii) $G(\sigma\, (x)\, \sigma(y))_{\alpha}\, k(\alpha)\, \theta(\sigma\tau(m))\, k(\alpha)\, G(\tau(d)\, \tau(z))_{\beta} = 0$

Proof

(i) Let us put x+ z for x in Lemma 2.11, branch (i)
$G(\sigma(x + z)\, \sigma\, (y))_{\alpha}\, k(\beta)\, \theta(\sigma\, \tau(m))\, k(\beta)\, G(\tau(y)\, \tau(x + z))_{\alpha} = 0$

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha + G(\sigma(z)\sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha + G(\sigma(z)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha = 0$

In light of Lemma (2.11)(i), the following is

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha + G(\sigma(z)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha = 0$

Consequently, we obtain

$G(\sigma(x),\sigma(y))_\alpha k(\beta)\theta(\sigma\tau(m))k(\beta)G(\tau(y),\tau(z))_\alpha k(\beta)\theta(\sigma\tau(m))k(\beta)G(\sigma(x),\sigma(y))_\alpha k(\beta)\theta(\sigma\tau(m))\ k(\beta)G(\tau(y)\ \tau(z))_\alpha = 0$

$= -G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)$

$G(\sigma(z)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha = 0$

Therefore, being Ḿ is prime,

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha = 0$ ……………(7)

Now, replacing $x + w$ for y in Lemma 2.11, branch (i), to result

$G(\sigma(x)\ \sigma(y+w))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y+w)\ \tau(x))_\alpha = 0$

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(w))_\alpha k(\beta)\theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(w))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha = 0$

Deriving from Lemma (2.11)(i), the result is

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(w))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha = 0$

Thus, we get:

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha = 0$

$= -G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ \ G(\sigma(x)\ \sigma(w))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha = 0$

Being Ḿ is prime , then:

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha = 0$ …………..(8)

Thus:

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y+w)\ \tau(x+z))_\alpha = 0$

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(y)\ \tau(z))_\alpha + G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(x))_\alpha + G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(z))_\alpha = 0$

Based on (7), (8) and also Lemma (2.10) (i), it leads to

$G(\sigma(x)\ \sigma(y))_\alpha k(\beta)\ \theta(\sigma\tau(m))\ k(\beta)\ G(\tau(w)\ \tau(z))_\alpha = 0$

(ii) If we put (β) instead of (α) in (i), this results (ii).

(iii) Putting $(\alpha + \beta)$ for (α) in (ii), to result

$G(\sigma(x)\ \sigma(y))_{\alpha+\beta} k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_{\alpha+\beta} = 0$

$G(\sigma(x)\ \sigma(y))_\alpha k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\alpha + G(\sigma\ (x)\ \sigma\ (y))_\alpha k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta + G(\sigma(x)\ \sigma(y))_\beta k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\alpha + G(\sigma(x)\ \sigma(y))_\beta k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta = 0$

By both (i) and (ii), we obtain

$G(\sigma(x)\ \sigma\ (y))_\alpha k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta + G(\sigma\ (x)\ \sigma\ (y))_\beta k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\alpha = 0$

Thus, the result will be

$G(\sigma(x)\ \sigma(y))_\alpha k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\sigma(x)\ \sigma(y))_\alpha k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta = 0$

$= -\ G(\sigma\ (x)\ \sigma\ (y))_\alpha k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha) G(\tau(x)\ \tau(y))_\beta k(\alpha)\ \theta(\sigma\ \tau(m))\ k(\alpha)\ G(\tau(w)$

$\tau(z))_\alpha = 0$

But Ḿ is prime , then:

$G(\sigma\ (x)\ \sigma\ (y))_\alpha k(\alpha)\ \theta(\sigma\tau(m))\ k(\alpha)\ G(\tau(w)\ \tau(z))_\beta = 0$

3. The Main Result

3.1. Theorem

Consider we have a Jordan (σ,τ)-k-homomorphism of a Γ-ring L to a prime Γ́-ring Ḿ, then we have (σ,τ)-k-homomorphism or (σ,τ)-k -anti homomorphism [6,7].

Proof

Consider θ Jordan (σ,τ)-k-homo-morphism of a Γ-ring L to a prime Γ́-ring Ḿ. Since Ḿ is prime Γ́-ring , then Lemma 2.12, (i) results:

$G(\sigma\ (x)\ \sigma\ (y))_\alpha k(\beta)\ \theta(\sigma\ \tau(m))\ k(\beta)\ G(\tau(w)\ \tau(z))_\alpha = 0$

But Ḿ is prime Γ́-ring, this means either $G(\sigma\ (x), \sigma\ (y))_\alpha = 0\}$ or $G(\sigma\ (w), \sigma\ (z))_\alpha = 0$, (x, y, z, w) ∈ l and α ∈ Γ.

When $G(\sigma\ (w), \sigma\ (z))_\alpha \neq 0$ (z, w) ∈ M and α ∈ Γ, then $G(\sigma\ (x), \sigma\ (y))_\alpha = 0$ }, x, y ∈ L and α ∈ Γ , hence we have θ is (σ,τ)-k-homomorphism.

Now, if $G(\sigma\ (w), \sigma\ (z))_\alpha = 0$, (z, w)∈ L and α ∈ Γ . This means θ is (σ,τ)-k-anti- homomorphism [7,8].

3.2. Propositionn

Assume that θ as a Jordan (σ,τ)-k-homo-morphism from a Γ-ring L to a 2-torsion free Γ́-ring Ḿ, satisfying $(x\alpha y\beta x = x\beta y\alpha x)$, x, y ∈ L and α, β ∈ Γ), $(x'\alpha'y'\beta'x' = x'\beta'y'\alpha'x')$, where x', y' ∈ Ḿ and α', β'∈ Γ́) , ($\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma\ \tau = \tau\ \sigma$) , then θ is Jordan triple (σ,τ)- k-homo-morphism [7,8].

Proof

Replace (y) by $(x\beta y + y\beta x)$ in definition 2.3:

$\theta(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = \theta(\ \sigma\ (x))k(\alpha)\theta(\tau(x\beta y + y\beta x)) +$

$\theta(\ \sigma\ (x\beta y + y\beta x))k(\alpha)\theta(\tau(x)) = \theta(\ \sigma\ (x))k(\alpha)\theta(\tau(x)\beta\tau(y) + \tau(y)\beta\tau(x)) + \theta(\sigma(x)\beta\ \sigma\ (y) + \sigma\ (y)\beta\sigma(x))k(\alpha)\theta(\tau(x)) = \theta(\ \sigma\ (x))k(\alpha)\theta(\sigma\ \tau(x))\ k(\beta)\ \theta(\tau^2(y)) + \theta(\sigma(x))\ k(\alpha)\ \theta(\sigma\tau(y))\ k(\beta)\ \theta(\tau^2(x)) + \theta(\sigma^2(x))\ k(\beta)\ \theta(\tau\sigma(y))\ k(\alpha)\ \theta(\tau(x)) + \theta(\sigma^2(y))\ k(\beta)\ \theta(\tau\sigma(x))\ k(\alpha)\ \theta(\tau(x))$

Since $(x'\alpha'y'\beta'x' = \beta'y'\alpha'x')$, a', b' ∈ Ḿ and α', β'∈ Γ́ ,

$\sigma^2 = \sigma$, $\tau^2 = \tau$ and $\sigma\tau = \tau\sigma$, we get:

$= \theta\sigma(x)\, k(\alpha)\, \theta\sigma\tau(x)\, k(\beta)\, \theta\tau(y) + 2\theta\sigma(x)\, k(\alpha)\, \theta\sigma\tau(y)\, k(\beta)\, \theta\tau(x) + \theta\sigma(y)\, k(\beta)\, \theta\sigma\tau(x) k(\alpha) \theta\tau(x)$ ………….(9)

On other hand:

$\theta(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = \theta(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)$

Since : $x\alpha y\beta x = x\beta y\alpha x$, where $x, y \in L$ and $\alpha, \beta \in \Gamma$, we obtain

$= \theta(x\alpha x\beta y + y\beta x\alpha x) + 2\theta(x\alpha y\beta x)$

$= \{\theta(\sigma(x))\, k(\alpha)\, \theta(\sigma\tau(x))\, k(\beta)\, \theta(\tau(y)) + \theta(\sigma(y))\, k(\beta)\, \theta(\sigma\tau(x))\, k(\alpha)\, \theta(\tau(x)) + 2\theta(x\alpha y\beta x)$ …………(10)

Compare (9) and (10), produces :

$2\theta(x\alpha y\beta x) = 2\theta(\sigma(x))\, k(\alpha)\, \theta(\sigma\tau(y)\, k(\beta)\, \theta(\tau(x))$

Since $\acute{M}$ 2-torsion free $\acute{\Gamma}$-ring, then θ will be Jordan triple (σ,τ)-k-homomorphism.

Its findings are a generalization of the classical theory of the Jordan homo-morphisms to the newer theory of (σ, τ)-k-homomorphisms of Γ-rings. The primeness and 2-torsion freeness conditions play an important role in the derivation of the main theorem. It is in this way that the mapping $k:\Gamma\rightarrow\Gamma^-$ can be introduced and more flexibility and applicability in non-commutative and generalized ring structures can be achieved. Studies into a Γ-ring have recently examined comparable mappings with respect to derivations and anti-derivations and so there seems to be an increasing interest in this field [7,8].

4. Conclusion

In this paper, we have defined and analyzed a number of new classes of homomorphisms in Γ-rings, such as (σ, τ)-k-homomorphisms and the Jordan versions of these homo-morphisms. The primary theorem demonstrates that, given the pre-requisites of natural conditions, any Jordan (σ, τ)-k-homomorphism to prime ring is homomorphism. The other case is anti-homo-morphism. This extends previous work and gives a comprehensive set of results on which to study mappings between Γ-rings. The current research can be advanced to examine such issues in other sets of rings or with weaker algebraic conditions.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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