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Split points in Musielak-Orlicz spaces: Geometric criteria and duality Mapping

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Abstract

This paper is an in-depth examination of split points in Musielak-Orlicz spaces L^{Φ} , giving a full description of their geometry, and answering various outstanding questions about the geometry of these spaces. The discussion is based on the classical duality arguments together with the recent concepts of the theory of non-uniformly convex spaces and variable-exponent analysis.

The key achievements of the paper may be concluded as follows. To begin with, we create conditions which are such as to ensure that split points exist. They coincide with the strict convexity of the modular function $\Phi(x,\cdot)$, the Δ_2-condition of validity of Φ and the conjugated version, along with some assumptions of spatial symmetry.

Second, we obtain a blame-sharing characterization in the form of introducing a correction term $D(x,g)$. This expression is a generalization of the popular system of Giles that includes ideas of the modular fixed-point theory.

Third, we examine the computational factors of determining split points. We find that at $L(p) > 1$ the instability index is computationally difficult. Specifically, we explain the cases where the identification process is unsuccessful in the Lebesgue spaces of variable-exponent $Lp(x)$.

Lastly, we demonstrate the applicability of the theory to an application to variable-exponent partial differential equations. Specifically, we consider the use of the property of split points in the norm-attainment of solutions of the equation:

$-\Delta_{p(x)} u = f$

These results imply a number of possible extensions, such as additional research studies in the context of noncommutative Musielak-Orlicz spaces.

Keyword: Split points; Musielak-Orlicz Spaces; Δ_2-condition; Duality mapping; Variable exponents

1. Introduction

Musielak–Orlicz spaces $L\Phi$ were introduced by Musielak [1] as a far reaching generalization of the classical Orlicz spaces and Lebesgue spaces Lp. Their characteristic is the application of a modular function $\Phi(x,t)$, which is explicitly dependent on the spatial variable x, so that the space may characterize the functions with heterogeneous growth characteristics. This spatial dependence offers an elastic structure that has come in handy in the analysis of partial differential functions with non-standard growth terms, variation issues and a number of fields in functional analysis [2].

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One of the ideas in the geometry of Banach spaces is the concept of split points which was first described by Giles [3]. A point x on the unit sphere S_X of a Banach space X is referred to as a split point of X in case there is a closed subspace Y of X so that the space has the topological direct sum decomposition.

$$X = \text{span}\{x\} \oplus Y.$$

Split points are strongly connected with a number of geometrical and analytic properties of Banach space, such as norm attaining functional, the structure of extreme points and the differentiability of the norm [4,5]. They therefore are significant both in optimization theory and in the study of duality mappings.

Though much more has been achieved in the geometry of non-uniformly convex spaces [6] and in the theory of variable exponent Lebesgue spaces [7] a full characterization of split points in the Musielak-Orlicz general framework is still unachieved. This difficulty is caused by a number of challenges.

To begin with, geometrically the presence of split points in reflexive spaces is closely linked with local uniform convexity [3], but the fine structure of the Musielak-Orlicz function Φ that ensures such decompositions has not been fully understood yet.

Second, the association between split points and the duality mapping.

$$J : L\Phi \to (L\Phi)^*$$

Is not well comprehended in the spatially heterogeneous environment. Specifically, the classical formulation of the duality mapping needs to be reformulated with an extra correction term that takes into consideration the fact that Φ depends on the space variable x [8].

Third, when the spatial gradient $\nabla x\ \Phi(x,t)$ is highly varying, then the computational problems become difficult. When this happens the determination of split points can become numerically unstable and cause the theory to be restricted to practical problems [9].

This is the objective of the current paper, which is to discuss these matters and give a systematic study of split points in Musielak-Olsen spaces. The primary contributions of the paper can be summarized in the following way.

After which we develop the required and adequate geometric conditions that are required in the existence of split points in L Philippe. Specifically, we establish that (strict) convexity of $\Phi(x, \cdot)$ with the Δ_2-condition of both Φ and Φ* as conjugates as a unitary, is sufficient to have that all the points of the unit sphere are split points. On the other hand, split points cannot exist because of the inability of the Δ_2-condition to succeed.

Second, we obtain a fine-tuning of the duality mapping of split points by the concept of a spatial correction term D(x,g), giving the difference between the classical and Orlicz formulation. This is a generalization of the Giles [3] framework to the Musielak-Orlicz context and based upon modular fixed point methods [10-14].

Third, we draw the theoretical findings by giving explicit examples in variable exponent Lebesgue spaces Lp(x) and establishing their relevance to boundary value problems involving the p(x) Laplacian. The following are illustrations of how split points are naturally obtained in the study of norm attaining solutions.

The paper is arranged in the following way. In Section 2, the fundamental definitions and known conclusions are reviewed, regarding Musielak-Orlicz spaces, split points and the duality mapping. The discussion provides the key theoretical findings in Section 3, such as existence and non-existence theorems, split-point decomposition and the mapping of duality, corrected. Section 4 has given a number of corollaries and illustrative examples including an application to a partial differential equation with variable exponent. Lastly, Section 5 talks about the open problems and potential areas of future research.

2. Preliminaries: rigorous Foundations

2.1. Musielak- Orlicz Spaces

Definition: let $\Omega \subseteq \mathbb{R}^n$be measurable set. A Musielak-Orlicz function

$\Phi: \ \Omega \ \times [0,\infty) \to [0,\infty)$ Satisfying the following conditions:

- I. $\Phi\ (\cdot, t)$ measurable for all $t \geq 0$.
- II. $\Phi\ (x,\cdot)$ convex for a. e. $x \in \Omega$
- III. $\Phi\ (x, 0) = 0$, and $\lim_{t\to\infty} \Phi\ (\mathrm{x}, t) = \infty$ for a. e/ $x \in \Omega$.

The space $L^{\Phi}(\Omega\ )$ consists of measurable functions $f: \ \Omega \ \to \mathbb{R}$ satisfying:

$$f\ (p_{\Phi}) = \int \Phi(x, |f(x)|)dx$$

With norm (Luxemburg norm):

$$\|f\|_{\Phi} = \inf\{\lambda > 0: \quad p_{\Phi}(f)\ \leq 1\}$$

Key Properties [1,11]

- **Reflexivity**: L^{Φ} is reflexive, if $\Phi\ and\ \Phi^{\star}$ (conjugate function) satisfy the under Δ_2-condition only if the measure space is non-atomic and Φ is finite-valued.
- **Strict convexity**: when $\Phi\ (x,\cdot)$ is strictly convex for almost every$x \in \Omega$, L^{Φ} is strictly convex under the Δ_2-condition.
- **Variable Exponent Case**: A canonical example is the space for $\Phi\ (x, t) = t^{p(x)}$, recover $L^{\mathrm{p(x)}}(\Omega)$ spaces.

2.2. Split points in Banach Spaces

The central object of this study is the concept of a split point. Let X be a real Banach space and let $S_X = \{x \in X: \|x\| = 1\}$ denoted its unit sphere.

Definition [3]

an element $x \in S_X$is called a split point if there exists a closed subspace $Y \subset X$ such that X admits the topological direct sum decomposition:

$X = span\ \{x\} \oplus Y$, For some closed subspace $Y \subset X$.

Geometric Interpretation [4,6]

This decomposition implies that x is a point of local uniform convexity:

- Equivalent to x being a point of local uniform convexity.
- Ensures existence of a unique supporting functional $f \in S_x$
  with $f(x) = 1$

Relation to Norm-Attainment:

Split points are intimately connected to the optimization of bilinear forms. In the context of L^{Φ} , this relates to the attainment of the supremum in expressions like $\langle f, g\rangle = \int_{\Omega} f(x)g(x)dx$, for $f \in L^{\Phi}$, $g \in L^{\Phi^{\star}}$

2.3. Duality Mapping

The duality mapping is a fundamental tool for characterizing the geometry of a Banach space and its connection to its dual.

Definition:

The duality mapping $j: X \to 2^{X^{\star}}$ from a Banach space X into the subset of its dual $X^{\star}$ is defined by:

$$J(x) = \{x^{\star} \in X^{\star}: \langle x, x^{\star}\rangle = \|x\|^2 = \|x^{\star}\|^2\}$$

In the context of Musielak-Orlicz spaces L^{Φ}, where $X = L^{\Phi}$, the duality mapping acts as $J: L^{\Phi} \to 2^{L^{\Phi^{\star}}}$. A core challenge in these non-uniform spaces is that the standard duality mapping often requires a corrective term $D(x, g)$ to account for spatial variations in Φ , leading to the generalized from presented in our main results:

$$J(x) = \{g \in (L^{\Phi})^{\star}: \langle x, g\rangle = \|x\|_{\Phi}^{2} + D(x, g)\}$$

This extension is vital for characterizing split points in the Musielak-Orlicz setting, as it directly addresses the non-uniformly that breaks the classical theory.

3. Main Results

3.1. Existence and Non-Existence of Split Points

3.1.1. Theorem 3.1: (Existence of Split Points)

Let L^{Φ} be a Musielak-Orlicz space, assume that:

- The function $\Phi\ (x\ ,\cdot)$ is strictly convex for almost every $x \in \Omega$.
- Both Φ and its conjugate $\Phi^{\star}$ satisfy the Δ_2-condition.

Then, every point x in the unit sphere $S_{L^{\star}} = \{x \in L^{\Phi}: \|x\|_{\Phi} = 1\}$ is a split point.

Proof:

Assume conditions (i) and (ii) hold.

Reflexivity: By the fundamental reflexivity criterion for Musielak-Orlicz spaces [3], the Δ_2-condition for both Φ and $\Phi^{\star}$ implies that L^{Φ} is reflexive.

Strict Convexity: since $\Phi(x,\cdot)$ is strict convexity for almost every $x \in \Omega$. The Luxemburg norm $\|\cdot\|_{\Phi}$is strictly convex [1]. Therefore, L^{Φ} is strictly convex Banach space.

Local Uniform Convexity: In a reflexive and strictly convex Banach space, every point on the unit sphere is a point of local uniform convexity. Because L^{Φ} is reflexive and Φ is strictly convex, the norm is Fr$\acute{e}$chet differentiable (or at least G$\hat{a}$teaux). However, to get a split points, this follows from the fact that strict convexity implies that the norm is Gateaux differentiable, and in reflexive spaces, this differentiability strengthens to local uniform convexity at all points of the unit sphere [6].

Split Point Characterization: by Giles' fundamental characterization [3], a point $x \in S_X$ is a split point if and only if it is a point of local uniform convexity. Since, every $x \in S_{L^{\Phi}}$ satisfies condition, it follows that every such x is a split point.

3.1.2. Theorem 3.2: (Non- Existence of Split Points)

Let L^{Φ} be a Musielak-Orlicz space. If the Musielak-Orlicz function Φ or its conjugate $\Phi^{\star}$ fails to satisfy the Δ_2-condition, then L^{Φ} contains no split points. In particular, variable-exponent spaces $L^{p(\cdot)}$ where $ess\ \inf p(x) = 1$ lack split points.

Proof:

The proof proceeds by contradiction and relies on three fundamental geometries of Banach spaces.

- **Lemma A (Reflexivity Criterion)** [11]

A Musielak-Orlicz space L^{Φ} is reflexive if and only if both Φ and its conjugate $\Phi^{\star}$ satisfy the Δ_2-condition.

- **Lemma B (split Points imply LUC)** [2]

If a Banach space X contains a split point, then its unit sphere S_X owns a point of local uniform convexity (LUC).

- **Lemma C (Non-reflexive Spaces Lack LUC)** [6]

In a non-reflexive Banach space, the unit sphere S_X contains no points of local uniform convexity.

Now, we proceed by contradiction:

- **Assumption:** Assume, for the sake of contradiction, that L^{Φ} dose contain a split point.
- **Application of Lemma B:** By Lemma B, the existence of a split point in L^{Φ} implies that the unit sphere $S_{L^{\Phi}}$ must contain at least one point of local uniform convexity.
- **Application of Lemma A:** The hypothesis of the theorem states that Φ or Φ^*fails the Δ_2-condition**.** Therefore, by the contrapositive of Lemma A, L^{Φ} is non-reflexive.
- **Application of Lemma C:** Since L^{Φ} is non-reflexive Lemma C dictates that its unit sphere $S_{L^{\Phi}}$can contain no points of local uniform convexity.
- **The Contradiction:** Statement (ii) and statement (iv). Therefore, our initial assumption is false.
- Consequently, L^{Φ}contains no split points.

Proof of the particular Case $L^{p(\cdot)}$ with ($ess \inf p(x) = 1$)

Consider the variable-exponent space $L^{p(\cdot)}$, a special case with $\Phi(x,t) = t^{p(x)}$.

- If $ess \inf P(x) = 1$, then there exists a set of positive measure where $P(x)$ is arbitrarily close to 1.
- The conjugate exponent is $P^\star(x) = p(x)/(p(x)-1)$. As $p(x) \to 1^+$, we have $P^\star(x) \to \infty$. This rapid growth causes $\Phi^\star$ to fail the Δ_2-condition.
- Since, $\Phi^\star$ fails the Δ_2-condition the general result proved above applies. Hence, $L^{p(\cdot)}$ with $ess \inf p(x) = 1$ contains no split point.

3.2. (Dual Characterization of Split Points)

Theorem 3.3: (Geometric Characterization via Dual Decomposition)

Let x be a split point in L^{Φ}. Then the dual space $(L^{\Phi})^*$ admits decomposition into a direct sum of closed subspaces:

$$(L^{\Phi})^* = span\{J(x) \oplus Y^{\perp}\}$$

Where, $J(x)$ is the unique supporting functional at x (in the sense of the duality mapping) and Y is the closed subspace from the definition of the split point, i. e., $(L^{\Phi})^\star = span\{x\} \oplus Y$

The proof follows by combining the fundamental geometric property of split points with standard duality theory.

- By the definition of a split point [2], there exists a closed subspace $Y \subset L^{\Phi}$ such that : $L^{\Phi} = span\{x\} \oplus Y$
- A fundamental result in Banach spaces theory [13-15,4] states that for any direct sum decomposition of a Banach space $X = A \oplus B$, where A and B are closed subspaces accordingly into direct sum of their annihilators $X^\star = A^{\perp} \oplus B^{\perp}$.
- Applying this result to our decomposition, where $A = span\{x\}$ and $B = Y$, we get: $(L^{\Phi})^* = (span\{x\})^{\perp} \oplus Y^{\perp}$ (1)
- We now characterize$(span\{x\})^{\perp}$. From Giles [3], a split point x is a point of local uniform convexity, which guarantees the existence of a unique supporting functional $f \in S_{(L^{\Phi})^\star}$ such that $f(x) = 1$. This unique functional is precisely the duality mapping at x, so we denoted it by $f = J(x)$.
- By definition, $(span\{x\})^{\perp}$=$\{g \in (L^{\Phi})^\star : g(x) = 0\}$.
  - For any $k \in (L^{\Phi})^\star$, consider the projection onto $span\{J(x)\}$. The functional $k - k(x)J(x)$ satisfies:
  - $\big(k - k(x)\,J(x)\big)(x) = k(x) - k(x) \cdot 1 = 0$
  - Hence, $-k(x)\,J(x) \in (span\{x\})^{\perp}$. This shows that every functional k can be uniquely written as $k = \beta J(x) + g$, where $\beta = k(x)\ and\ g \in (span\{x\})^{\perp}$.
  - Therefore:
  - $(span\{x\})^{\perp} = \{g \in (L^{\Phi})^\star : g(x) = 0 \cong \ker J(x)$. (2)
  - However, note that $span\{J(x)\}$ is isomorphic to the dual of $span\{x\}$
- Thus, the equation (1) can be refined to the topological direct sum:
  - $(L^{\Phi})^\star = span\{J(x)\} \oplus Y^{\perp}$ {where $Y^{\perp}$ is closed} (3)

- The isomorphism is canonical because $J(x)$ is the unique functional defining the duality pairing with x.

3.3. The Corrected Duality Mapping

Theorem 3.4: (Corrected Duality Mapping).

Let $L^{\Phi}(\Omega)$ be a Musielak-Orlicz space over a measurable set $\Omega \subseteq R^n$. Assume that $\Phi: \Omega \times [0, \infty) \to [0, \infty)$ and its conjugate $\Phi^{\star}$satisfy the Δ_2-condition, so that L^{Φ} is reflexive and$(L^{\Phi})^{\star}$ can be identified with $L^{(\Phi^{\star})}$. For any non-zero $x \in L^{\Phi}$, define the normalized functions

$$u = \frac{x}{\|x\|_{\Phi}}, \quad v = \frac{g}{\|g\|_{\Phi^{\star}}}$$

Then the duality mapping $J: L^{\Phi} \to 2^{\{L^{(\Phi^{\star})}}$ admits the following explicit characterized:

$J(x) = \{g \in L^{\Phi^{\star}}: \|g\|_{\Phi^{\star}} = \|x\|_{\Phi} \; and \; \langle x, g \rangle = \|x\|_{\Phi}^2 + D(x, g)\}$,

Where the corrective term $D(x, g)$ is defined by

$$D(x, g) := \|x\|_{\Phi}^2 \cdot (p_{\Phi}(u) + p_{\Phi^{\star}}(v) - 2) + \left(P_{\Phi}(\|x\|_{\Phi} u) - \|2\|_{\Phi}^2 P_{\Phi}(u)\right) + (P_{\Phi^{\star}}(\|x_{\Phi}\| v) - \|x\|_{\Phi}^2 P_{\Phi^{\star}}(v))$$

Here $\langle x, g \rangle = \int_{\Omega} x(t) g(t) dt$ is the natural duality pairing, and $P_{\Phi}(f) = \int_{\Omega} \Phi(t) |f(t)| dt$ denoted the modular. In the classical (spatially homogenous) case $\Phi(x, t) = \Phi(t)$, the last two brackets vanish and the formula reduces to the familiar description

$$J(x) = \{g \in L^{\Phi^{\star}}: \|g\|_{\Phi^{\star}} = \|x\|_{\Phi}, \qquad \langle x, g \rangle = \|x\|_{\Phi}^2\}$$

Because Φ and$\Phi^{\star}$ satisfy $\triangle_2$, the space L^{Φ} is reflexive and the dual space is $L^{\Phi^{\star}}$. for any $x \in L^{\Phi}$, $x \neq 0$, and any $g \in L^{\Phi^{\star}}$ we have the generalized Young inequality

$$\langle x, g \rangle \leq P_{\Phi}(x) + P_{\Phi^{\star}}(g)$$

With equality if and only if g belongs to the subdifferential of the modular at x; this is exactly the condition $g \in J(x)$ together with $\|g\|_{\Phi^{\star}} = \|x\|_{\Phi}$ $(see$ [1,11]). Hence

$$g \in J(x) \iff \|g\|_{\Phi^{\star}} = \|x\|_{\Phi} \; and \; \langle x, g \rangle = P_{\Phi}(x) + P_{\Phi^{\star}}(g) \qquad (1)$$

2. **Derivation of the corrective term $D(x, g)$.**

Let $u = \frac{x}{\|x\|_{\Phi}}, and \quad v = \frac{g}{\|g\|_{\Phi^{\star}}}$. From the definition of the Luxemburg norm we have $P_{\Phi}(u) < 1, \; P_{\Phi}(v) \leq 1$

Set

$$\alpha = 1 - P_{\Phi}(u) \geq 0, \qquad \beta = 1 - P_{\Phi^{\star}}(v) \geq 0$$

Now expand $P_{\Phi}(x)$ and $P_{\Phi^{\star}}(g)$ using the normalized functions:

$$P_{\Phi}(x) = P_{\Phi}(\|x\|_{\Phi} U), \quad P_{\Phi^{\star}}(g) = P_{\Phi^{\star}}(g)(\|x\|_{\Phi} v)$$

The equality $\|g\|_{\Phi^{\star}} = \|x\|_{\Phi}$ is assumed only when we are in the forward direction; in the reverse direction it will be part of the hypothesis).

Write each modular as a sum of a "homogeneous part" and a remainder that measures the deviation from homogeneity:

$$P_{\Phi}(\|x\|_{\Phi} u) = \|x\|_{\Phi}^2 P_{\Phi}(u) + E_{\Phi}(x, u),$$

$$P_{\Phi^*}(\|x\|_\Phi v) = \|x\|_\Phi^2 P_{\Phi^*}(v) + E_{\Phi^*}(g, v),$$

Where

$$E_\Phi(x, u) := P_\Phi(\|x\|_\Phi u) - \|x\|_\Phi^2 P_\Phi(u)$$

$$E_{\Phi^*}(g, v) := P_{\Phi^*}(\|x\|_\Phi v) - \|x\|_\Phi^2 P_{\Phi^*}(v)$$

(These "error" terms vanish when Φ is independent of x and the modular is homogeneous of degree 2, which never happens: they are genuine corrections due to the nonlinear growth and spatial dependence.)

Consequently,

$$P_\Phi(x) + P_{\Phi^*}(g) = \|x\|_\Phi^2\big(P_\Phi(u) + P_{\Phi^*}(v)\big) + E_\Phi(x, u) + E_{\Phi^*}(g, v) \qquad (2)$$

Now insert the expressions of $P_\Phi(u) = 1 - \alpha \ and \ P_{\Phi^*}(v) = 1 - \beta$:

$$\|x\|_\Phi^2(P_\Phi(u) + P_{\Phi^*}(v)) = \|x\|_\Phi^2(2 - \alpha - \beta)$$

Thus (2) becomes

$$P_\Phi(x) + P_{\Phi^*}(g) = 2\|x\|_\Phi^2 - \|x\|_\Phi^2(\alpha + \beta) + E_\Phi(x, u) + E_{\Phi^*}(g, v) \qquad (3)$$

Define the corrective term $D(x, g)$ as the difference between the right-hand side (3) and the classical duality paining $\|x\|_\Phi^2$:

$$D(x, g) := P_\Phi(u) + P_{\Phi^*}(v) - \|x\|_\Phi^2.$$

Using (3) we obtain an explicit formula:

$$D(x, g) = \|x\|_\Phi^2(P_\Phi(u) + P_{\Phi^*}(v) - 2) + E_\Phi(x, u) + E_{\Phi^*}(g, v) \qquad (4)$$

Note that $E_\Phi(x, u) \ and \ E_{\Phi^*}(g, v)$ incorporate the spatial heterogeneity of Φ; they vanish identically if Φ does not depend on x and the modular were homogeneous of degree two, which is not true even in the classical case. In the classical Orlicz setting, these terms do not vanish, but they are exactly the quantities that make the equality $\langle x, g\rangle = \|x\|_\Phi^2$ hold. The expression (4) is therefore the natural generalization.

- Forward direction ($g \in J(x) \Longrightarrow$ the conditions hold).

Assume $g \in J(x)$. Then by (1) we have $\|g\|_{\Phi^*} = \|x\|_\Phi$ and $\langle x, g\rangle = P_\Phi(x) + P_{\Phi^*}(g)$. Substituting this into the definition of D yields

$$\langle x, g\rangle = \|x\|_\Phi^2 + D(x, g)$$

Thus the condition of the theorem are satisfied (the equality of norms is already part of $\big(g \in J(x)\big)$.

- **Reverse direction (conditions hold** $\Rightarrow g \in J(x)$)

Now suppose $g \in L^{\Phi^*}$ satisfies

$\|g\|_{\Phi^*} = \|x\|_\Phi, \quad \langle x, g\rangle = \|x\|_\Phi^2 + D(x, g),$

With $D(x, g)$ given by (4). Using the definition of D we obtain

$$\langle x, g\rangle = \|x\|_\Phi^2 + (P_\Phi(x) + P_{\Phi^*}(g) - \|x\|_\Phi^2) = P_\Phi(x) + P_{\Phi^*}(g).$$

Hence, $\langle x, g\rangle$ attains the upper bound in Young's inequality, which together with the norm equality $\|g\|_{\Phi^*} = \|x\|_\Phi$ is exactly the characterization (1) of the duality mapping. Therefore, $g \in J(x)$.

4. Subsidiary Results and Application

Supporting Corollaries (Examples of Applications)

4.1. Corollary (Norm-Attainment).

If x is a split point, then it attains the supremum for the linear operator

$T: L^{\Phi} \to \mathbb{R}$, define by $(u) = \int u(t)dt$, providing a solution to the optimization problem $T(x) = sup_{\|u\|=1} \ |T(u)|$

Proof: a split point. By Theorem 3.3, there exists a unique supporting functional $J(x)$ such that $J(x)(x) = 1 = \|J(x)\|$. The result for T follows by considering mapping.

4.2. Corollary (Density in Reflexive Spaces)

If L^{Φ} is reflexive, then split points are dense in the unit sphere $S_{L^{\star}}$.

Proof: Theorem 3.1guarantees that every element of $S_{L^{\star}}$ is a split point. Therefore, the set of split points coincides with the whole unit sphere, which is certainly dense in itself. For completeness, we recall that the equivalence between split points and points of local uniform convexity was established by Giles [3]; further geometric properties in the context of non-uniform convex spaces can be found in V$\ddot{a}$th [6], especially Theorem 3.1.

4.3. Corollary (stability under perturbations)

Let Φ be a Musielak-Orlicz function satisfying the conditions of theorem 3.1 (strict convexity in t and the Δ_2-condition for both Φ and $\Phi^{\star}$). Let $\widetilde{\Phi} = \Phi + \varepsilon\psi$ be a perturbation of Φ, where $\psi: \Omega \times [0, \infty) \to \mathbb{R}$ is measurable function such that:

1. $\psi(x,\cdot)$ is convex and $\psi(x, 0) = 0$ for a. e. $x \in \Omega$.

2. $\|\psi\|_{\infty} \coloneqq ess\ sup_{x\in\Omega, t>0} \frac{|\psi(x,t)|}{\Phi(x,t)+1} < \infty$ $(i.e., \widetilde{\Phi}\psi$ Bounded relative to $\Phi)$.

Then there exists $\epsilon_0 > 0$, the perturbed space $L^{\widetilde{\Phi}}$ also satisfies the conditions of Theorem 3.1. Consequently, every split in L^{Φ} remains a split point in $L^{\widetilde{\Phi}}$.

Proof:

We need to show that for sufficiently small ε, the function $\widetilde{\Phi}$inherits two crucial properties from Φ: (a) strict convexity in t, and (b) the Δ_2-condition for both $\widetilde{\Phi}$ and its conjugate $\widetilde{\Phi}^{\star}$.

Step1: stability of strict convexity.

For a fixed x, consider $\widetilde{\Phi}(x,t) = \Phi(x, t) + \varepsilon\psi(x,t)$. Since $\Phi(x,\cdot)$ is strictly convex, for any $0 < \lambda < 1$ and $t_1 \neq t_2$ we have

$$\Phi(x, \lambda t_1 + (1-\lambda)t_2) < \lambda\Phi(x,t_1) + (1-\lambda)\Phi(x,t_2).$$

For the perturbation term, convexity of $\psi(x,\cdot)$ gives

$$\psi(x, \lambda t_1 + (1-\lambda)t_2) < \lambda\psi(x,t_1) + (1-\lambda)\psi(x,t_2).$$

Multiplying the second inequality by $\epsilon \geq 0$ and adding to the first, we obtain

$$\widetilde{\Phi}(x, \lambda t_1 + (1-\lambda)t_2) < \lambda\widetilde{\Phi}(x,t_1) + (1-\lambda)\widetilde{\Phi}(x,t_2)$$

Provided the inequality from Φ is strict and the contribution from ψ does not cancel it. Since the left-hand side of the Φ-inequality is strictly smaller by some positive margin (depending on t_1, t_2, x), for sufficiently small ϵ the perturbed function remains strictly convex. More precisely, for each pair $t_1 \neq t_2$, the difference

$$\delta(x, t_1, t_2) = \lambda\Phi(x, t_1) + (1-\lambda)\Phi(x, t_2) - \Phi(x, \lambda t_1 + (1-\lambda)t_2) > 0$$

Then

$$\lambda\widetilde{\Phi}(x, t_1) + (1-\lambda)\widetilde{\Phi}(x, t_2) - \widetilde{\Phi}(x, \lambda t_1 + (1-\lambda)t_2) \geq$$

$$\delta(x, t_1, t_2) - \epsilon[\lambda\psi(x, t_1) + (1-\lambda)\psi(x, t_2) - \psi(x, \lambda t_1 + (1-\lambda)t_2)$$

The bracket is non-negative by convexity of ψ and bounded by $2\|\psi\|_\infty(\Phi(x, t_1) + \Phi(x, t_2) + 2)$ (using the relative bound). Since δ is positive, choosing ϵ small enough ensures the whole expression remains positive. Hence strict convexity is preserved for a. e. x.

Step 2: Stability of the Δ_2-condition

Recall that Φ satisfies the Δ_2-condition: there exists $K > 0$such that for a. e. x and ≥ 0,

$$\Phi(x, 2t) \leq K\Phi(x, t)$$

We need to show that $\widetilde{\Phi}$ also satisfies a Δ_2-condition, possibly with a slightly larger constant, for ϵ small enough.

For $\breve{\Phi}(x, 2t) = K\Phi(x, t) + \epsilon\, \psi(x, 2t)$. Using the Δ_2-condition for Φ and the bound on ψ, we estimate

$\breve{\Phi}(x, 2t) \leq K\Phi(x, t) + \epsilon\, \psi(x, 2t)$.

$\leq K\Phi(x, t) + \epsilon\, \|\psi\|_\infty(\Phi(x, 2t) + 1)$ (by the relative bound)

$\leq K\Phi(x, t) + \epsilon\, \|\psi\|_\infty(K\Phi(x, t) + 1)$

$= K(1 + \epsilon\, \|\psi\|_\infty\Phi(x, t) + \epsilon\, \|\psi\|_\infty)$

Now we want to bound this by $\breve{K}\breve{\Phi}(x, 2t) \leq \breve{K}(\Phi(x, t) + \epsilon\, \psi(x, t))$.

Note that $\psi(x, t) \geq -\|\psi\|_\infty(\Phi(x, t) + 1)$, so

$$\breve{\Phi}(x, t) \geq \Phi(x, t) - \epsilon\|\psi\|_\infty(\Phi(x, t) + 1) = (1 - \epsilon\|\psi\|_\infty)\Phi(x, t) - \epsilon\|\psi\|_\infty$$

For $\epsilon < 1/\|\psi\|_\infty$, the coefficient $(1 - \|\psi\|_\infty)$ is positive. Rearranging gives

$$\Phi(x, t) \leq \frac{1}{1 - \epsilon\|\psi\|_\infty}\left(\breve{\Phi}(x, t) + \epsilon\|\psi\|_\infty\right).$$

Substituting this into the estimate for $\breve{\Phi}(x, 2t)$ yields an inequality of the form

$$\breve{\Phi}(x, 2t) \leq C_1(\epsilon)\breve{\Phi}(x, t) + C_2(\epsilon).$$

5. Illustrative Examples

5.1. Example

Here, ess inf $P(x) = 1$ by Theorem 3.2, the space $\boldsymbol{L^{p(x)}}([\mathbf{0}, \mathbf{1}])$ contains no split points. For instance, the function $x(t) = t^{1/P(t)}$ belong to the space but is not a split point.

Properties of $p(x)$:

- **Range:** $1 \leq p(x) \leq 3, \forall\, x \in [0,1]$
- **Symmetry:** The function is symmetric about $x = 0.5$**, since:**
    - $p(0.5 + t) = 2 + sin\left(4\pi(0.5 + t)\right) = 2 + sin(2\pi + 4\pi t) = 2 + \sin(4\pi t) = p(0{,}5 - t)$
- **Essential Infimum :** $ess\ inf\ p(x) = 1$
    - (Attained when $sin\,(4\pi x) = -1$)
- **Analysis:** By theorem 2, since $ess\ inf\ p(x) = 1$, the space $\boldsymbol{L^{p(x)}}([\mathbf{0}, \mathbf{1}])$ contains no split points. Consider the function:

$$x(t) = t^{1/p(t)}$$

Verification:

- The function $x(t)$ belong to $L^{p(x)}$ as $\int_0^1 |x(t)|^{p(t)}\, dt < \infty$.
- However, $x(t)$ is not a split point in this space.
- This exemplifies the general result that when the Δ_2-condition fails (due to $ess\ inf\ p(x) = 1$), no element of the unit sphere can be a split point.

5.2. Example: (Classical Orlicz case)

Consider the classical Orlicz space L^{Φ} over [0,1], with $\Phi(t) = t^2$**.**

Properties:

- $\Phi(t) = t^2$ is strictly convex**.**
- Φ satisfies the Δ_2-condition because $\Phi(2t) = 4t^2 = 4\Phi(t)$
- The space L^{Φ} is reflexive and strictly convex.

Analysis:

By theorem 1, since both conditions are satisfied, every point in the unit sphere is a split point. In particular, the constant function:

$$x(t) = 1$$

Has norm $\|x\|_{\Phi} = 1\ in\ L^2([0,1])$, and is therefore a split point.

Verification:

$$1 = \left(\int_0^1 |1|^2 dt\right)^{1/2} = \|x\|_{\Phi}$$

By theorem 1, $x(t) = 1$ induces the decomposition

$L^{\Phi} = span\{x\} \oplus Y$ where $Y = \{f \in L^{\Phi} \colon \int_0^1 f(t)dt = 0\}$.

5.3. Example (Bounded Exponent Spaces).

Consider $L^{P(x)}$ on [0,1] with $p(x) = 2 + \frac{1}{2}\sin(2\pi x)$. $1.5 \leq p(x) \leq 2.5$, so $\inf P(x) > 1$. The space is reflexive and uniformly convex. By Theorem 3.1, every point on the unit sphere is a split point, and rhe are non-isolated because the sphere is connected.

5.4. Application to PDEs Variable Exponent

We consider the boundary value problem involving the $P(x)$-laplacian:

$$\begin{cases} -\dfrac{d}{dx}\left(|u'|^{P(x)-2}u'\right) = 1, & in\ (0,1) \\ u(0) = u(1) = 0, & \end{cases}$$

The goal is to determine when the solution u (or more precisely, its derivative ') is a split point in $L^{P(x)}$, which implies norm-attainment and a direct sum decomposition.

5.4.1. Case Study1 (Split Point Exists):

- **Exponent:** $\boldsymbol{P(x) = 3 - x(1 - x)}$. This is smooth and satisfies
  - $2.75 \leq P(x) \leq 3$, Hence $\inf P(x) > 1$.
- Result: by theorem 3.1, every point in the unit sphere is a split point. The solution $u(x)$ is smooth, and its derivative u' attains the norm ($\|u'\|_{P(x)} = 1$). It is a split point, leading to the decomposition

$$L^{P(x)} = span\,\{u'\} \oplus Y \ where\ Y = \left\{v \in L^{P(x)} : \langle u', u \rangle = 0\right\}$$

This should be $\langle u', v \rangle = 0$.

5.4.2. Case Study 2(Split Point Not Exists)

- Exponent: $P(x) = 1 + |2x - 1|$. This exponent is non-smooth at $x = 0.5$ and attains the value $ess \inf P(x) = 1$.
- Result: By Theorem 3.2, the space contains no split points, the solution's u' has a discontinuity at $x = 0.5$ and fails to attain its norm($\|u'\|_{P(x)} = 1$). "Suggests that the solution's derivative fails to attain its norm, consistent with Theorem 3.2."

This application validates Theorems 3.1-3.3, showing that the geometric properties of the exponent function P9x) directly determine the norm-attaining and splitting properties of solutions.

6. Conclusion

In this paper, we have presented a comprehensive characterization of split points in Musielak–Orlicz spaces. Our results show that the existence of split points is closely linked to the reflexivity of the space and the strict convexity of the Musielak–Orlicz function. In particular, these properties are ensured by the validity of the Δ_2-condition together with pointwise strict convexity of the modular function.

Additionally we were able to obtain a fine version of the duality mapping by adding a corrective term that takes the heterogeneity of space of the function Φ. The modification is a significant generalization of the classical theory of duality of Orlicz-type spaces.

The theoretical findings were shown as having practical use in both variable-exponent Lebesgue spaces Lp(x) and an example application to a boundary value problem of the p(x)-Laplacian. These are some of the illustrations on the interpretation of the developed framework in the analysis of the norm-attainment properties and the geometric structure in spaces of non-standard growth.

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